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A SPECTRAL THEORETIC APPROACH TO FAULT ANALYSIS IN LINEAR SEQUENTIAL CIRCUITS

S. Sangani

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In this paper the problem of fault analysis in linear and affine sequential circuits is treated. These classes of systems provide for the treatment of several linear and nonlinear faults common in digital circuitry. The solution to the linear and affine sequential circuit problem is obtained via the development of a spectral theory for such systems over finite fields. A stepwise fault analysis procedure for this problem class is presented along with many examples illustrating the advantages that memory provides in digital fault analysis.

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ABSTRACT

In this paper the problem of fault analysis in Linear and Affine Sequential Circuits is treated. These classes of systems provide for the treatment of several linear and nonlinear faults common in digital circuitry. The solution to the linear and affine sequential circuit fault analysis problem is obtained via the development of a spectral theory for such systems over finite fields. A stepwise fault analysis procedure for this problem class is presented along with many examples illustrating the advantages that memory provides in digital fault analysis.

CHAPTER 1

INTRODUCTION

1.1 Introduction to Fault Analysis:

The rapid progress of modern integrated circuit technology has made it possible to manufacture many new digital circuits with complexity orders of magnitude greater than the circuits of the preintegration period. A semiconductor "chip" which is just the size of a fingernail, may have many individual circuit elements embeded in it. Moreover, a number of such chips may be combined together to construct a large circuit. This renders the structure of the circuit more complex which in turn makes maintenance difficult. If the circuit does not behave in the manner it is supposed to, the faulty element in the circuit must be located. Techniques used for locating the faulty part or parts of a circuit are termed as fault analysis techniques.

Historically, fault analysis techniques for analog and digital circuits have developed independently. The reader is referred to the bibliographies compiled by Rault[1,2] for a review of the literature in the two areas. Digital fault analysis techniques are mainly combinatorial in nature. i.e., one tests each component of the circuit by applying a family of test inputs to that circuit. If one obtains the expected response from the circuit for each test input, the circuit is operating properly and so are all of its components. On the other hand, if the circuit fails to operate correctly for one or more test inputs, the faulty component(s) may be isolated by determining the set of components which are exercised by exactly that set of input test signals. In the case of analog circuits, one has the advantage of testing the circuit by exponential signals of different frequencies. Hence the

gain between various circuit test points at each of several different frequencies may be measured. Then by some means, an equation solver or optimization routine, the circuit parameters are determined which will yield these gains. If the resultant component parameters are in the operating range, the corresponding component can be assumed to be working correctly, whereas if a computed component parameter deviates significantly from its operating range, the corresponding components may be assumed faulty.

Fault analysis techniques for digital and analog circuits with memory differs from each other. In digital circuits,[3] due to combinatorial approach, test complexity increases exponentially with memory. (Since the number of entries in a circuit truth table increases exponentially.) In case of dynamical analog circuits, however, just the opposite is true. One, here, uses the multifrequency testing techniques in which one may determine the gain between a pair of test points at several different frequencies simultaneously from a single test signal. Thus total number of test inputs required for testing dynamical analog circuits are less than for a memoryless circuit of a similar complexity.

1.2 Fault Analysis of Analog Circuits:[4,5,6,7,8,9,10]

Analog fault analysis is heavily pedicated on spectral theoretic techniques formulated in a frequency domain setting. In the area of fault analysis of analog circuits a number of recent papers[4,5,11] have given considerable attention to the component connection model of a large scale system. Via the component connection model fault analysis is based on a system modeling technique wherein the observable system behavior is expressed explicitly as a function of the internal responses. In the component connection model, one assumes that the ith

component is characterized by an operator equation (usually expressed in the frequency domain),

$$b_i = Z_i a_i \qquad i = 1,2,-,-,-,-,n \tag{1.1}$$

mapping the component input vector, a_i, into its output vector, b_i. Although in actual practice one normally works with the n separate component equations, notationally (1.1) may be combined into the single equation

$$b = Z a \tag{1.2}$$

where $b = \text{col.}(b_i)$

$a = \text{col.}(a_i)$

$Z = \text{diag}\ (Z_i)$

Since the connection elements (adders, scalers, etc.) are all algebraic, the connection model is represented by a set of linear algebraic equations of the form,

$$\left[\begin{array}{c} a \\ \hline y \end{array}\right] = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] \left[\begin{array}{c} b \\ \hline u \end{array}\right] \tag{1.3}$$

where y is the overall system output and u is the input vector for the overall system which are related by the overall system operator, S, by

$$y = S u \tag{1.4}$$

Upon combining equations (1.2), (1.3), and (1.4), one obtains the equation,

$$S = L_{22} + L_{21}(1-ZL_{11})^{-1}ZL_{12} = f(Z) \tag{1.5}$$

relating the overall system operator to the composite component operator. Since the function, f, is entirely determined by the fixed connection matrices, it has been termed the connection function.[5] Now, if Z and S are characterized in the frequency domain, one obtains,

$$S(\omega) = f(Z(\omega)) \tag{1.6}$$

since the connections are independent of frequency.

Finally, in many practical systems one can write the component matrix, $Z(\omega)$, in the form

$$Z(\omega) = E(\omega)CF(\omega) \tag{1.7}$$

where $E(\omega)$ and $F(\omega)$ are frequency dependent matrices determined by the component types but not their actual values and C is a frequency independent matrix of component values (i.e. for an inductor the $j\omega$ goes in E or F and L goes in the C matrix). Such a viewpoint is quite reasonable for the fault analysis problem wherein one can reasonably assume that the component types remain fixed and all faults manifest themselves as changes in component values. We then combine the known E and F matrices into the connection function and characterize our system by

$$S(\omega) = f(E(\omega)CF(\omega)) = f_{\omega}(C) \tag{1.8}$$

Equation (1.8) is just the right form in which to study the fault analysis problem for if we make the (standard) hypothesis that all faults take the form of errors in C with the connections and component types fixed, i.e. f_{ω} is fixed, then one merely measures $S(\omega)$ at some frequency and solves equation (1.8) for C. Unfortunately, solution of these non-linear equations is equivalent to left invertibility of the matrix

$$K = L_{12}^{T} \otimes L_{21} \tag{1.9}$$

which requires that the system have a large number of test points.

If equation (1.8) is not soluble one has two alternatives; either add more test points (which increases hardware costs) or use several test frequencies (which increases software costs). The former is straightforward and may be carried out algorithmically by adding additional test points in such a manner that the additional rows added to the matrix K will render its columns linearly independent. From a practical

point of view, however, it is preferable to use additional test frequencies at the same test points in which case one must solve the set of simultaneous equations

$$
\begin{aligned}
S(\omega_1) &= f_{\omega_1}(C) \\
S(\omega_2) &= f_{\omega_2}(C) \\
&\vdots \\
S(\omega_k) &= f_{\omega_k}(C)
\end{aligned}
\tag{1.10}
$$

Equations (1.10) illustrate the essence of the multifrequency testing idea in that the additional test frequencies give us more equations for the same number of unknowns (which is due to our assumption that the component variations with frequency are known and non-faulty). Moreover, the technique exploits the system dynamics for if the components were non-dynamical then f_{ω} would be independent of ω and the equations obtained from additional frequencies would be redundant. As such, for analog circuits multifrequency testing makes it easier to perform fault analysis in a dynamical system than in a non-dynamical system (in the sense that fewer test points are required).

1.3 Fault Analysis of Digital Circuits [3,12]

Traditionally it has been believed that fault analysis for sequential circuits (digital circuits with memory) is much harder than for combinatorial circuits, the above described results for the analog case suggest that the opposite might be true. That is, one may formulate a technique for the fault analysis of digital circuits which parallels the multifrequency testing techniques of analog fault analysis by exploiting the circuit dynamics so as to require fewer test inputs for a sequential circuit than for a combinatorial circuit of similar complex-

ity. Since a Linear Sequential Circuit can be viewed as a linear operator on a sequence space, a perfectly valid spectral theory for the Linear Sequential Circuit may be formulated. The resulting spectral theory parallels the steady state frequency domain theory for analog circuits and thus may be used to formulate a fault analysis procedure for Linear Sequential Circuits which closely parallel the multifrequency testing techniques for analog circuits.

The sequential circuits are defined over a finite field. These finite fields are denoted by GF(p) where p is a prime and its nth order extension is denoted by $GF(p^n)$. A detailed review of finite fields and an algorithm for generating elements of $GF(p^n)$ is given in Appendix A. All these extension fields lie in the algebraic closure[13] of a finite field.

In the Chapter II, the spectral theory for Linear Sequential Circuits is formulated. For this purpose a Linear Sequential Circuit is mathematically described by a pair of difference equations over a finite field. Rather than interpreting this set of equations as the traditional initial value problem, it is interpreted as a central value problem of finding two sided state and output sequences. i.e., for the positive and negative values of time.

The delays used in the sequential circuit are interpreted as predictors. i.e., The next state value of the sequential circuit is the delayed present state of the sequential circuit. The corresponding definition of the D-transform is given in Appendix B.

The entire mathematics outlined in the previous section for the analog case goes through if one interprets the vectors; a, b, u and y; as sequences taking their values in the finite field rather than real

valued functions and assumes that the entries in the L matrices take their value in the same finite field.

In the case of sequential circuits if one deals with D-transform rather than laplace transform, equation (1.1) through equation (1.6) goes through i.e.,

$$S(D) = f(Z(D)) = f((G)F(D)) = f_D(G) \qquad (1.11)$$

where f_D is a nonlinear function which is entirely determined by component dynamics and the connections relating component parameter values to the systems input and output. Here, the implication that f_D is known and non-faulty is that all faults occur in the scalers, G, with memory elements and connection good. In particular, this implies a linear system fails linearly and hence one may include "stuck-on-zero", "open" and "short circuit" faults but not "stuck-on-one" faults which are nonlinear. "Stuck-on-one" faults are, however, included when a Linear Sequential Circuit is generalized to the case of affine circuits which fail affinely. The latter generalization also permits a slight generalization of the traditional Linear Sequential Circuit case by allowing NOT gates and bias sources in addition to the usual Linear Sequential Circuit components.

Finally, the analog of exponential test functions for the case of sequential circuits are the sequences, $\{u^e\}$, of the form,

$$\{u_k^e\} = \{e^k\}, \qquad k = 0, \pm 1, \pm 2, \pm 3, \cdots$$

where e is an element of the algebraic closure of a finite field.

The input sequence $\{u_k\} = \{e^k\}$ yields the equality,

$$y_k = S(e)\, e^k$$

for an appropriate initial state.

These exponential sequences "live" in an extension of the space on

which the actual circuit is defined and therefore can not be physically implemented. This is precisely the same phenomena which occurs in analog circuits where one must test with the real valued input $\mathrm{Sin}(\omega t)$ to obtain information about the complex valued $e^{j\omega t}$. Therefore, in Chapter II, a technique for computing S(e) directly from the system impulse response is derived.

Once S(e) is known, one can write the set of equations parallel to those used for analog fault analysis for several e's in the algebraic closure of a finite field. i.e.,

$$\begin{aligned} S(e_1) &= f_{e_1}(G) \\ S(e_2) &= f_{e_2}(G) \\ &\vdots \\ S(e_k) &= f_{e_k}(G) \end{aligned} \tag{1.12}$$

As in the analog case it may be possible to solve this set of simultaneous equations even though no single equation has a unique solution. Thus it is possible to exploit the dynamics in a Linear Sequential Circuit in a similar manner to that used in the analog case, so as to simplify the fault analysis procedure.

In Chapter II, the required spectral theory is formulated and also a technique for computing S(e) directly from the impulse response is derived. In Chapter III, a formula for computing S(e) from the Linear Sequential Circuit component parameters and the connection matrices is obtained. As such, one may test a circuit with an impulsive input, compute S(e) therefrom and then compute the component parameters by inverting this latter formula. This inversion process can be formulated as the solution of a set of polynomial equations in several variables

via the "term expansion algorithm"[14] which is described in Chapter III. A family of illustrative examples appears in Chapter IV and the generalization to affine circuits is presented in Chapter V.

CHAPTER II

SPECTRAL THEORY - LINEAR SEQUENTIAL CIRCUITS

2.1 "Two sided" Linear Sequential Circuits:

The characteristics of a Linear Sequential Circuit[15] (LSC) are depicted in Figure 2.1.

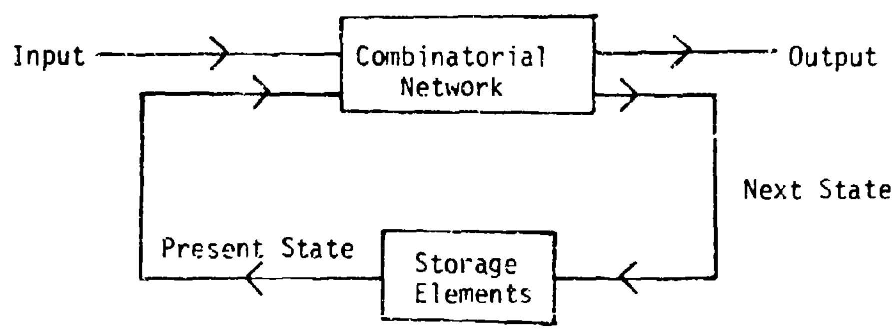

Figure 2.1: Block Diagram Representation of a Linear Sequential Circuit

Essentially, a Linear Sequential Circuit consists of storage elements and combinatorial logic. An input sequence applied to an LSC results in an output sequence, whose present value is a linear function of the present input value and the present state. The present state in turn is a linear function of past states and past inputs. Hence, a Linear Sequential Circuit can be viewed as a linear operator on a sequence space.

Mathematically, a "two sided" Linear Sequential Circuit over a finite field is represented by a set of difference equations,

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Ju_k \qquad -\infty < k < \infty \\
x_0 &= \underline{x}
\end{aligned}
\tag{2.1}
$$

Where the state sequence X, the output sequence Y and the input sequence U are respectively s, m and r dimensional vectors over the finite field. x_k, y_k and u_k are respectively the state, the output and the input sequence values at time k for the LSC. The matrices A, B, C and J are finite field valued and constant with dimensions consistent with x_k, y_k and u_k.

Traditionally one interprets equation (2.1) as an initial value problem in which one desires to find "one-sided" sequences; x_k and y_k, $k \geq 0$ satisfying equation (2.1) for a given sequence u_k and initial state, $x_0 = \underline{x}$. One interpretes equation (2.1) as a central value problem wherein one seeks "two sided" sequences; x_k and y_k, $-\infty < k < \infty$ satisfying (2.1) for a given sequence u_k and central value $x_0 = \underline{x}$. Unlike the case for LSC's defined over "one sided" sequences, this central value problem for difference equations (2.1) does not admit a unique solution for all $x_0 = \underline{x}$. In this chapter, the theory of existence and uniqueness of solutions to the central value problem is developed and a viable spectral theory for the difference equations (2.1) which closely parallels steady state frequency domain theory for continuous linear systems is formulated. Such a spectral theory provides a way for determining faults in a Linear Sequential Circuit.

The existence and uniqueness of solutions to the homogeneous version of (2.1) which is,

$$x_{k+1} = Ax_k$$
$$y_k = Cx_k \qquad -\infty < k < \infty \qquad (2.2)$$
$$x_0 = \underline{x}$$

will now be presented in the first two lemmas. The existence and uniqueness of solutions to (2.1) will then be established in the third lemma.

Lemma 1: Let A be a linear transformation on a vector space X, then X has a "fitting decomposition"[16] given by,

$$X = f_0 \oplus f_1$$

Where $f_0 = \{z | A^t z = 0,\ z \in X,\ t \geq T \text{ for some } T\}$ and $f_1 = R(A^t) = R(A^{t+1}) = \ldots\ldots\ldots\ldots \quad t \geq T$.

The subspaces f_0 and f_1 are respectively the "Fitting Null" and the "Fitting 1" components of X. Furthermore, the restriction of A to f_0, $A_0 : f_0 \to f_0$ is nilpotent i.e., $A_0^{\ell} = 0$ for some ℓ, and the restriction of A to f_1, $A_1 : f_1 \to f_1$ is invertible on f_1.

The proof of this lemma is given in the literature,[16] but for completeness and convenience is included here.

Proof:

Let $x \in R(A^2)$, then $x = A[Ay]$ for some y. Which implies that $x \in R(A)$ and hence $R(A^2) \subseteq R(A)$. It then follows that rank $(A^2) \leq$ rank (A). Similarly, one can inductively show that, $R(A) \supseteq R(A^2) \supseteq R(A^3) \supseteq \ldots\ldots \supseteq R(A^r)$ and rank $(A^r) \leq$ rank $(A^{r-1}) \leq \ldots\ldots \leq$ rank $(A^2) \leq$ rank (A). Note that as r gets larger, rank (A^r) will eventually remain constant since it is bounded from below by zero. Thus there exists an r such that

$$\text{rank}(A^r) = \text{rank}(A^{r+1}) = \ldots\ldots \text{ or}$$

$$R(A^r) = R(A^{r+1}) = \ldots\ldots = f_1.$$

Define $\beta_i = \{z | A^i z = 0, z \in X\}$ (2.3A)

Then $\beta_1 = \{z | Az = 0 = A^2 z = A[\beta_1]\}$, so one has $\beta_1 \subseteq \beta_2$, similarly one can inductively show that $\beta_1 \subseteq \beta_2 \subseteq \beta_3 \subseteq \ldots\ldots$ Then there exists s for which the null space defined by equation (2.3A) will attain some constant value. It follows that $\beta_s = \beta_{s+1} = \ldots\ldots = f_0$.

Let $t = \max(r,s)$. Then $f_0 = \beta_t$ and $f_1 = R(A^t)$.

Any $x \in X$ can be written as $x = (x - A^t y) + A^t y$, $A^t y \in f_1$ for some $y \in X$. We also know that $A^t x = A^{2t} y$ i.e., $A^t(x - A^t y) = 0$ which implies $(x - A^t y) \in f_0$. Hence we have $x = f_0 + f_1$.

Let $z \in f_0 \cap f_1$. Then $z = A^t w$ for $w \in X$, since $A^t w \in f_0$, $A^t w = 0$ implying $z = 0 = f_0 \cap f_1$, thus $X = f_0 \oplus f_1$.

Since $f_1 = R(A^r) = R(A^{r+1}) = \ldots\ldots = f_1 A$, hence A is surjective in f_1. Since $f_0 = \beta_t$, $A^t = 0$ in f_0 then A is an isomorphism on f_1 i.e., $A_1 : f_1 \to f_1$ is invertible on f_1.

Lemma 2: (Homogeneous Case)

(a) The Equation $x_{k+1} = Ax_k, \; -\infty < k < \infty, \; x_0 = \underline{x}$ (2.3)

defined over a finite field has a solution if and only if $x_0 = \underline{x} \in f_1$. In this case the solution is unique, takes its values in f_1 for all k and is given by $x_k = A_1{}^k \underline{x}, \; -\infty < k < \infty.$ (2.4)

(b) The Equation $x_{k+1} = Ax_k, \; y_k = Cx_k, \; -\infty < k < \infty$ (2.5)

$x_0 = \underline{x}$ defined over a finite field has a solution if and only if $\underline{x} \in f_1$. In this case the solution is unique, takes its values in $C(f_1)$ for all k and is given by $y_k = CA_1{}^k \underline{x}, \; -\infty < k < \infty.$ (2.6)

Proof:

(a) The equation $x_{k+1} = Ax_k$, $-\infty < k < \infty$ can also be written in the form, $x_k = A^T x_{k-T} \in R(A^T) \in f_1$ (2.7)

The condition that $\underline{x} \in f_1$ follows from the fact that in any solution $\underline{x} = x_0$ can be written in the form,

$$\underline{x} = x_0 = A^T x_{-T} \in R(A^T) = f_1$$

The claim is $x_k = A_1{}^k \underline{x}$, $x_0 = \underline{x} \in f_1$ (recall that the inverse of A_1 exists), $-\infty < k < \infty$ is a solution to (2.3). To prove this show that (2.4) satisfies (2.3).

$$\begin{aligned} Ax_k &= A\, A_1{}^k \underline{x} \\ &= A_1 A_1{}^k \underline{x} && \text{[A can be replaced by } A_1 \\ &= A_1{}^{k+1} \underline{x} && \text{since } A_1{}^k \underline{x} \in f_1] \\ &= x_{k+1} \end{aligned}$$

Uniqueness of the solution can be proven by contradiction. Let there be two solutions $x_{k_1} \in f_1$ and $x_{k_2} \in f_1$. Such that $x_{01} = x_{02} = \underline{x}$.

Then we have $\underline{x} = A_1{}^{-k} x_{k_1} = A_1{}^{-k} x_{k_2}$, $-\infty < k < \infty$ which yields $x_{k_1} = x_{k_2}$.

(b) The condition $\underline{x} \in f_1$ follows as in (a). The converse can be proved by constructing y_k. Multiplying both sides of equation (2.4) by C, one obtains

$$Cx_k = CA_1{}^k \underline{x} \in C\,(f_1) \text{ since } A_1{}^k \underline{x} \in f_1 \tag{2.8}$$

but comparing equations (2.5) and (2.8), one obtains that

$$y_k = CA_1^{\,k}\underline{x} \in C(f_1).$$

Uniqueness follows from the uniqueness of $A_1^{\,k}\underline{x}$.

The following examples illustrate Lemma 2:

Example 2.1:

Consider homogeneous LSC defined over GF(2) characterized by the following set of equations.

$$\begin{bmatrix} x_{1(k+1)} \\ x_{2(k+1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} \tag{2.9}$$

$$y_k = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_{1k} \\ x_{2k} \end{bmatrix} \tag{2.10}$$

with $x_0 = \underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in f_1$.

Clearly $x_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in f_1 \quad \forall k$ and $y_k = 1 \quad \forall k$ i.e., $y_k \in C(f_1)$.

Now consider $x_0 = \underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in f_1$, which yields,

$\{x_k\} = \{\ldots\ldots, \begin{matrix}0\\1\end{matrix}, \begin{matrix}1\\1\end{matrix}, \begin{matrix}0\\1\end{matrix}, \begin{matrix}1\\1\end{matrix}, \begin{matrix}0\\1\end{matrix}, \begin{matrix}1\\1\end{matrix}, \ldots\ldots\}$, i.e.,

$x_k \in f_1$ for all k and the output sequence, $\{y_k\}$, with $y_0 = 1$, is

$\{y_k\} = \{\ldots\ldots, 1, 0, 1, 0, 1, 0, \ldots\ldots\}$

i.e., $y_k \in C(f_1)$ for all k.

In the above example, note that the uniqueness of $\{y_k\}$ is solely determined by the "central value" of the state, $x_0 = \underline{x}$.

In the following example, two distinct central values, $x_0 = \underline{x}$,

one in f_1 and the other not in f_1 are considered. It is then shown that the solution exists for $\underline{x} \in f_1$ but does not exist for $\underline{x}$ not in f_1.

Example 2.2:

Consider a homogeneous LSC defined over GF(2) characterized by

$$\begin{bmatrix} x_{1(k+1)} \\ \\ x_{2(k+1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1k} \\ \\ x_{2k} \end{bmatrix}$$

Clearly,

$$R(A) = R(A^2) = R(A^3) = \ldots\ldots = f_1.$$

Consider

$$\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varepsilon\ f_1$$

Then $A_1 : f_1 \to f_1$ where $A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

With central value $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, one obtains,

$$x_k = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \varepsilon\ f_1 \qquad \text{for all } k$$

Now let $\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ not in f_1. Then one obtains $\hat{}_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ not in f_1.

Lemma 3: (Non Homogeneous Case)

Consider the equations, $x_{k+1} = Ax_k + Bu_k$, $-\infty < k < \infty$, defined over a finite field and let u_k be a periodic sequence. Then there exists $\underline{x}$ such that the equation has a periodic solution, $\{x_k\}$, with $x_0 = \underline{x}$.

Proof:

It is well known that when a finite state machine is driven by a one-sided ultimately periodic input sequence, the output sequence (and

therefore state sequence) produced by the machine exists for any initial condition and is ultimately periodic.[17]

Let $\{\tilde{u}_k\}$, k = 0, 1, 2, ... be the "one-sided" ultimately periodic input sequence constructed from a two sided periodic sequence $\{u_k\}$ via $\tilde{u}_k = u_k$, $k \geq 0$. Let $\{\tilde{x}_k\} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{NT}, \tilde{x}_{NT+1}, \ldots, \tilde{x}_{MT-1}, \ldots\}$ (where x_0 is any initial state) be the resultant "one-sided" ultimately periodic state sequence produced by $\{\tilde{u}_k\}$. Without loss of generality one may assume that the values $\tilde{x}_{NT}$, $\tilde{x}_{NT+1}, \ldots, \tilde{x}_{MT-1}$, $M > N$, constitute one period of the periodic portion of $\{\tilde{x}_k\}$ where T is the period of $\{u_k\}$. Next define a periodic sequence $\{y_k\}$ which coincides with $\{\tilde{x}_k\}$ for $NT \leq k \leq MT-1$ i.e., $x_k = \tilde{x}_r$ where r is the unique integer $r = k + q(M-N)T$ and $NT \leq r \leq MT-1$, q is an integer (positive, zero or negative). It may be shown that the sequence $\{x_k\}$ satisfies the nonhomogeneous equation.

Case 1: $(NT \leq r \leq MT-2)$

Substituting $r = k + q(M-N)T$ in the equation $\tilde{x}_{r+1} = A\tilde{x}_r + Bu_r$ yields

$$\begin{aligned} \tilde{x}_{k+1+q(MT-NT)} &= A\tilde{x}_{k+q(MT-NT)} + Bu_{k+q(MT-NT)} \\ &= A\tilde{x}_{k+q(MT-NT)} + Bu_k \end{aligned}$$

Which is

$$x_{k+1} = Ax_k + Bu_k$$

Case 2: $(r = MT-1)$

Substituting $r = MT-1$ in the equation $\tilde{x}_{r+1} = A\tilde{x}_r + Bu_r$ yields,

$$\tilde{x}_{MT} = A\tilde{x}_{MT-1} + Bu_{MT-1}\text{, since } u_{MT-1} = u_{NT-1}$$

one obtains,

$$x_{NT} = Ax_{NT-1} + Bu_{NT-1}$$

or $\quad x_{k+1} = Ax_k + Bu_k \quad$ for $\quad k = NT-1$.

Finally note the uniqueness of a periodic portion of $\{\tilde{x}_k\}$ i.e., $\tilde{x}_{NT}, \tilde{x}_{NT+1}, \ldots\ldots, \tilde{x}_{MT-1}$ where T is the period of $\{u_k\}$, implies the uniqueness of $\{x_k\}$ satisfying $x_{k+1} = Ax_k + Bu_k$, $-\infty < k < \infty$.

The following example illustrates Lemma 3:

Example 2.3:

Consider an LSC defined over $GF(2^2)$ and characterized by the set of equations,

$$\begin{bmatrix} x_{1(k+1)} \\ x_{2(k+1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1(k)} \\ x_{2(k)} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \qquad (2.12)$$

Let $\{u_k\} = \{1, \alpha, 1+\alpha, \alpha, 1+\alpha, \ldots\ldots\}$

and,

$$\underline{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which give

$$\{\tilde{x}_k\} = \left\{ \begin{matrix} \tilde{x}_0 & \tilde{x}_1 & \tilde{x}_2 & \tilde{x}_3 & \tilde{x}_4 & \tilde{x}_5 & \tilde{x}_6 & \\ 0 & 0 & 1 & 1+\alpha & 0 & 1 & 1+\alpha & \\ 0 & 1 & \alpha & 1+\alpha & 1 & \alpha & 1+\alpha & \ldots\ldots \end{matrix} \right\}$$

Consider one periodic portion of $\{\tilde{x}_k\}$ such that $\tilde{x}_3 = \tilde{x}_{NT}$

Then

$$\{x_k\} = \left\{ \ldots\ldots \begin{matrix} 1 \\ \alpha \end{matrix}, \begin{matrix} 1 \\ \alpha \end{matrix}, \begin{matrix} 1+\alpha \\ 1+\alpha \end{matrix}, \begin{matrix} 0 \\ 1 \end{matrix}, \begin{matrix} 1 \\ \alpha \end{matrix}, \begin{matrix} 1+\alpha \\ 1+\alpha \end{matrix}, \ldots\ldots \right\}$$

where $x_0 = \begin{bmatrix} 1+\alpha \\ 1+\alpha \end{bmatrix}$ is the solution to equation (2.12).

2.2 Spectral Theory:

For an operator M on a vector space, one says that a scalar, λ, is

said to be an eigenvalue if there exists a non-zero vector U such that

$$MU = \lambda U \tag{2.13}$$

and one says that U is the corresponding eigenvector. Although this definition is traditionally associated with real or complex matrices on a finite-dimensional space, the defining equality for eigenvectors and eigenvalues also holds for arbitrary operator on an abstract vector space. In particular, one can use the above equality to define the eigenvalues and eigenvectors of a single input - single output LSC by viewing it as a linear operator on the infinite dimensional vector space of "two-sided" sequences with values in GF(p). In this the defining equality becomes,

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ \lambda U_k &= Cx_k + Ju_k \end{aligned} \tag{2.14}$$

Here $\{u_k\}$ is an eigensequence taking its values in a finite field and the eigen values in general, takes their value in the algebraic closure of GF(p). This is easily verified by observing that the eigen values for a matrix, M, over GF(p) are the zeros of the polynomial det(λI-M). Although the coefficients of this polynomial are in GF(p), these zeros of the polynomial may lie in its algebraic closure. The algebraic closure of a finite field plays essentially the same role for an LSC defined over that field as complex numbers do for an analog circuit defined over the real field. The minor problems arising due to the use of the algebraic closure are discussed in the next section.

For the periodic input $\{u_k\} = \{e^k\}$, $k = 0, \pm 1, \pm 2, \ldots$ (2.15)

where e is an element of the algebraic closure of the finite field over which an LSC is defined. Define a "transfer function" in the

usual manner,

$$S(e) = J + C\,(eI-A)^{-1}\,B \tag{2.16}$$

Here, S(e) is a rational function in e with coefficients in a finite field, whose poles and zeros are well defined elements of the algebraic closure of the finite field. For each e (except for the poles of S(e)), the sequence $\{u_k\} = \{e^k\}$, $-\infty < k < \infty$ is an eigen sequence of the LSC with eigenvalue S(e). The eigenvalues of the LSC are elements of the algebraic closure and the eigensequences take their values in the algebraic closure just as eigenvalues and eigenfunctions, for a system defined over the real field are complex. The fundamentally important fact here is that the eigensequence is independent of the LSC under study and hence is assured to be an eigensequence for a faulty circuit even if the fault is unknown.

Theorem 1: Let an LSC be characterized by

$$x_{k+1} = Ax_k + Bu_k \tag{2.1A}$$

$$-\infty < k < \infty$$

$$y_k = Cx_k + Ju_k \tag{2.1B}$$

over a finite field. Then for each e in the algebraic closure of the finite field for which $S(e) = J + C\,(Ie-A)^{-1}\,B$ is defined, there exists an initial condition x_0 for (2.1) such that the sequence $\{u_k\} = \{e^k\}$, $k = 0, \pm 1, \pm 2, \ldots\ldots$ is an eigensequence for the LSC with this initial condition and eigenvalue is S(e).

Proof:

Since $\{u_k\} = \{e^{+k}\}$, $k = 0, \pm 1, \pm 2, \ldots\ldots$ is a periodic sequence, the existence of a solution to (2.1) is guaranteed by Lemma 3. The existence of an initial condition such that $\{u_k\} = \{e^k\}$, $k = 0, \pm 1, \pm 2, \ldots\ldots$ is an eigensequence with the eigenvalue S(e) can be proven by

substitution as follows:

The claim is that $x_k \doteq (Ie-A)^{-1} Be^k$ (2.17)

is a unique solution to $x_{k+1} = Ax_k + Bu_k$, $-\infty < k < \infty$ (2.18)

with an input sequence $\{u_k\} = \{e^k\}$, $k = 0, \pm 1, \pm 2, \ldots\ldots$ such that $x_0 = (Ie-A)^{-1} B$. To show this, substitute (2.17) in (2.18)

$$Ax_k + Bu_k = A[(Ie-A)^{-1} Be^k] + Be^k$$

$$= (Ie-A)^{-1} Be^{k+1}$$

$$= x_{k+1} .$$

Uniqueness of x_k can be shown be contradiction, let there be two solutions x_k' and x_k'' to equation (2.18) such that $x_0' - x_0'' = (Ie-A)^{-1} B$. Then one obtains

$$x_{k+1}' = Ax_k' + Bu_k \quad (2.19)$$

$$x_{k+1}'' = Ax_k'' + Bu_k \quad (2.20)$$

Subtracting (2.20) from (2.19)

$$(x_{k+1}' - x_{k+1}'') = A(x_k' - x_k'') \quad (2.21)$$

Let $x_k' - x_k'' = \tilde{x}_k$ then (2.21) becomes,

$$\tilde{x}_{k+1} = A\tilde{x}_k \qquad \tilde{x}_0 = x_0' - x_0'' = 0$$

Then from Lemma (-) one gets

$$\tilde{x}_k = 0$$

which implies,

$$x_k' = x_k''$$

This shows $x_k = (Ie-A)^{-1} Be^k$ is the unique solution to equation (2.1A) such that $x_0 = \underline{x} = (Ie-A)^{-1} B$.

Substituting (2.17) into (2.16) yields ·

$$y_k = Cx_k + Ju_k$$
$$= [C(Ie-A)^{-1} B + J]\, e^k$$
$$= S(e)\, e^k \qquad (2.22)$$

Uniqueness of y_k follows from the uniqueness of x_k $\forall k$.

The essence of Theorem 1 is that it allows one to interpret the "transfer function" of an LSC as a function such as is done for continuous time systems over the real field rather than as an abstract operator as is usually done for LSC's. Thus, it is a function which identifies an eigenvalue with its eigenvector. In the case of multiple input -multiple output, the above arguments go through with the matrix $S(e) = J + C(Ie-A)^{-1} B$ interpreted as a matrix of eigenvalues.

Example 2.4:

Consider the single input-single output Linear Sequential Circuit shown in Figure 2.2 and described by the equations

$$\begin{aligned} x_{k+1} &= x_k + u_k \\ y_k &= x_k \end{aligned} \qquad -\infty < k < \infty \qquad (2.23)$$

defined over GF(2).

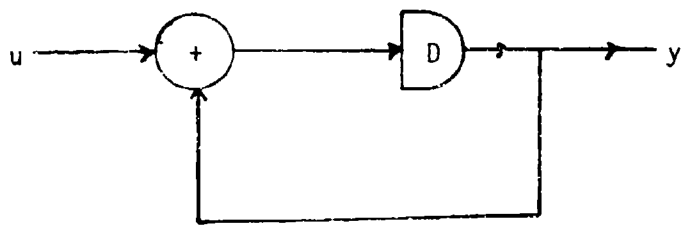

Figure 2.2: Figure for Example 2.4

Substituting the values of A, B, C, J, into (2.16) yields

$$S(e) = \frac{1}{1 + e} \qquad (2.24)$$

Consider the extension field GF(2^2) whose elements are (0, 1, α,

$\alpha + 1 = \alpha^2$) and let $e = \alpha$, then k. $\{u_k\} = \{e^k\} = \{\alpha^k\}$, i.e., $u_0 = 1$ and $\{u_k\} = \{. . . . 1, \alpha, 1 + \alpha, 1, \alpha, 1 + \alpha, 1, \alpha, 1 + \alpha, ...\}$ which produces an output sequence, $y_k = S(e)e^k = S(\alpha)\alpha^k$, i.e., $\{y_k\} = \{. . . . \alpha, \alpha^2, 1, \alpha, \alpha^2, 1, \alpha, \alpha^2, 1, ...\}$ where $y_0 = \alpha$. Now letting $e = \alpha + 1$, $\{u_k\} = \{\alpha + 1\}^k$ produces an output sequence $\{y_k\} = \{. . . . \alpha^2, \alpha, 1, \alpha^2, \alpha, 1, \alpha^2, \alpha, 1, ...\}$ where $y_0 = \alpha^2$.

In the above example, the eigensequence $\{u_k\} = \{e^k\}$ is ssociated with an eigenvalue $S(e) = \frac{1}{1 + e}$. This example shows that by choosing different elements from the extension field of GF(2), different output sequences are generated. Also note that in this case the state and the output sequences will be the same.

2.3 Computing S(e) from the Impulse Response:

Using input signals taking their values in the algebraic closure of a finite field presents two problems. First, one must be able to do computation in the extension field and, second, one must actually test the system with signals that take their values in the original field since the physical system is not capable of accepting inputs from the extension field. The first problem can be solved without much difficulty since $GF(p^m)$ can be represented as a field of m^{th} degree polynomials with coefficients in GF(p).[13] Here, addition is the usual polynomial addition and multiplication is the usual polynomial multiplication modulo of the $(m + 1)^{st}$ order irreducible polynomial.

The second problem can be solved by simply measuring the impulse response of the system and then computing the "exponential" response.

Since the goal here is to be able to find the relationship between the zero state impulse response; $\{h_k\}$, $k \geq 0$, and the matrix transfer function $S(e)$, one can assume that the input-output relation-

ship of the LSC is characterized by a minimal (A, B, C, J) realization, which characterizes the actual physical system under study. Since such a realization is controllable it can always be driven from the zero state at time $k = k_o$ to any desired state at time $k = 0$ by a sequence of inputs b_{-j}, $k_o \leq j < 0$ provided $k_o \leq -\delta$ where δ is the degree of the LSC.

The relationship between the "exponential" response of a single input - single output LSC and its zero state impulse response is given by the following theorem. The results of this theorem can easily be generalized for the case of multiple input - multiple output LSC's.

Theorem 2: Let $\{h_k\}$, $k \geq 0$ be the zero state impulse response of a single input - single output LSC and $S(e) = J + C(Ie-A)^{-1} B$ be its transfer function. The "exponential response" $S(e)e^k$ to an input sequence $\{u_k\} = \{e^k\}$ is given by,

$$S(e)e^k = \sum_{j=1}^{k_o} h_{j+k}\, b_{-j} + \sum_{i=-1}^{k-1} h_{i+1}\, e^{k-i-1} \quad , \; k > 0 \qquad (2.25)$$

The b_{-j}'s are (unknown) inputs that drive the system from the zero state at time $k = k_o$ to $x_o = \underline{x}$ (possibly non-zero) at $k = 0$.

Proof:

$$\text{Define} \quad \{\underline{e}^k\} \triangleq \begin{cases} \{e^k\} & k \leq -1 \\ \{0\} & k > -1 \end{cases} \qquad (2.26)$$

$$\{\bar{e}^{\,k}\} \triangleq \begin{cases} \{0\} & k \leq -1 \\ \{e^k\} & k > -1 \end{cases} \qquad (2.27)$$

Then $\{e^k\} = \{\underline{e}^k\} + \{\bar{e}^{\,k}\}$, $k = 0, \pm 1, \pm 2, \ldots\ldots$ Let y_k be the response of the LSC to $\{u_k\} = \{e^k\}$. Let $\underline{y}_k$ be the response of the LSC due to $\{u_k\} = \{\underline{e}^k\}$. Let $\bar{y}_k$ be the response of the LSC due to $\{u_k\} = \{\bar{e}^{\,k}\}$.

It follows firectly from the defining equations for the minimal realization of the LSC that

$$x_k = A^k \underline{x} + \sum_{i=0}^{k-1} A^i B e^{k-i-1} \tag{2.29}$$

and

$$y_k = CA^k \underline{x} + \sum_{i=0}^{k-1} CA^i B e^{k-i-1} + J e^k \tag{2.30}$$

Now let b_{-j}, $k_o < -j < 0$ be a sequence of inputs that drives the minimal realization from the zero state at time $k = k_o$ to $\underline{x}$ at time $k = 0$. Such a sequence exists for a minimal realization if $k_o < -\delta$, where δ is the dimension of the minimal realization.

It follows that

$$x_{-k_o} = 0$$

$$x_{-(k_o-1)} = Bb_{-n}$$

$$x_{-(k_o-2)} = ABb_{-n} + Bb_{-(n-1)}$$

$$x_{-(k_o-3)} = A^2Bb_{-n} + ABb_{-(n-1)} + Bb_{-(n-2)}$$

$$\vdots$$

$$\underline{x} = x_o = x_{-(k_o-k_o)} = [B \;\; AB \;\; \ldots \;\; A^{k_o-2}B \;\; A^{k_o-1}B] \begin{bmatrix} b_{-1} \\ b_{-2} \\ \vdots \\ b_{-(k_o)} \end{bmatrix}$$

which gives

$$CA^k\underline{x} = [CA^kB \quad CA^{k+1}B \; . \; . \; . \; . \; . \; CA^{k+k_o-2}B \quad CA^{k+k_o-1}B] \begin{bmatrix} b_{-1} \\ b_{-2} \\ \vdots \\ b_{-(k_o-1)} \\ b_{-k_o} \end{bmatrix} \tag{2.31}$$

The values of the impulse response $\{h_k\}$ are given in terms of the LSC minimal realization description by,

$$\begin{aligned} h_o &= J \\ h_1 &= CB \\ h_2 &= CAB \\ &\vdots \\ h_k &= CA^{k-1}B \quad , \quad k > 0 \end{aligned} \tag{2.32}$$

Substituting (2.32) in (2.31) yields,

$$CA^k\underline{x} = [h_{k+1} \quad h_{k+2} \; . \; . \; . \; . \; . \; h_{k+k_o-2} \quad h_{k+k_o-1}] \begin{bmatrix} b_{-1} \\ b_{-2} \\ \vdots \\ b_{-(k_o-1)} \\ b_{-k_o} \end{bmatrix}$$

$$= \sum_{j=1}^{k_o} h_{j+k}\, b_{-j} \tag{2.33}$$

Substituting (2.33) into (2.30) yields,

$$y_k = \sum_{j=1}^{k_o} h_{j+k}\, b_{-j} + \sum_{i=1}^{k-1} CA^i\, B\, e^{k-i-1} + J\, e^k$$

Utilizing (2.32) yields,

$$y_k = \sum_{j=1}^{k_o} h_{j+k}\, b_{-j} + \sum_{i=0}^{k-1} e^{k-i-1} + h_o e^k$$

$$= \sum_{j=1}^{k_o} h_{j+k}\, b_{-j} + \sum_{i=-1}^{k-1} h_{i+1}\, e^{k-i-1} \tag{2.34}$$

From (2 2) the "exponential response" is given by

$$y_k = S(e)e^k$$

Thus from (2.34), it follows that,

$$S(e)e^k = \sum_{j=1}^{k_o} h_{j+k}\, b_{-j} + \sum_{i=-1}^{k-1} h_{i+1}\, e^{k-i-1}$$

In c s where the LSC is not in the zero state, the zero state impulse res se can be determined as follows: Observe the response of the LSC due to an input sequence $\{u_k\} = \{0\}$. When this response becomes periodi , apply an impulse sequence to the LSC. The zero state impulse sponse is then given by the difference of two responses i.e., responses obtained after and before the impulse sequence is applied.

To apply Theorem 2, one observes that Equation (2.25) is linear in all the unknowns ($S(e)$ and the b_{-j}, $k_o \leq -j \leq 0$) hence by writing the equation for $k = 1, 2, \ldots, k_o+1$, one may set up a k_o+1 by k_o+1 matrix equation which may be solved for $S(e)$ in terms of the measured values of h_k, $0 < k < 2k_o+1$. In the case of multiple input - multiple output LSC's, $S(e)$ is a matrix of eigenvalues and the impulse response is also a matrix. The procedure for determining $S(e)$ in the single input - single output case can easily be generalized for determining the impulse response matrix and (in turn) the matrix $S(e)$.

The following example illustrates Theorem 1.

Example 2.5:

Consider the LSC defined in Example 2.3. The zero state impulse response is given by

$$h_0 = 0,\ h_1 = h_2 = h_3 = \ .\ .\ .\ .\ .\ .\ . = 1,$$

from Theorem 1

$$S(e)e^k = \sum_{j=1}^{k_0} h_{j+1}\, b_{-j} + \sum_{i=-1}^{k-1} h_{i+1}\, e^{k-i-1}$$

Let $e = \alpha \in GF(2^2)$, $k_0 = 1$.

For $k = 1$, $S(\alpha)\alpha = b_{-1} + 1$ (2.35)

and for $k = 2$, $S(\alpha)\alpha^2 = b_{-1} + \alpha + 1$ (2.36)

Solve (2.35) and (2.36) simultaneously, one obtains,

$$S(\alpha) = \alpha = \frac{1}{1 + \alpha} \tag{2.37}$$

Now let $e = \alpha^2 = 1 + \alpha \in GF(2^2)$,

for $k = 1$, $S(\alpha^2)(\alpha^2)^1 = b_{-1} + 1$ (2.38)

for $k = 2$, $S(\alpha^2)(\alpha^2)^2 = b_{-1} + \alpha^2 + 1$ (2.39)

Solving (2.38) and (2.39) simultaneously, one obtains,

$$S(\alpha^2) = \frac{1}{1 + \alpha^2} \tag{2.40}$$

Equations, (2.37) and (2.40), could be verified by substituting the corresponding value of e in equation (2.24).

CHAPTER III

FAULT ANALYSIS - LINEAR SEQUENTIAL CIRCUITS

3.1 Component Connection Model:

A Linear Sequential Circuit is usually characterized by an input-output state model for the purpose of fault analysis; wherein one attempts to collect input-output data to determine faulty components within the LSC. The fault analysis algorithm developed in this chapter uses the component connection which relates input-output behavior directly to component parameters rather than the state.

The component connection model was first intuitively used by Prasad and Traboth[18] and has been used by several other investigators in the area of fault analysis in analog circuits.[4,5,14]

The primary reason in choosing the component connection model for fault analysis is that it is so heavily algebraic that it unifies various graphical and diagramatic connection theories and at the same time smooths the transition from mathematical model to computer algorithm.

In the sequel the digital version of the component connection model is developed which is essentially the same as in the analog case except for minor changes. The development is repeated here for the sake of completeness and for demonstrating the interpretation of terms used in the model for the digital case.

A mathematical interpretation of a system is: a mapping from a set of inputs to a set of outputs i.e., an input-output relation. Letting u and y represent system input and output sequences, respectively, with values in a finite field or its algebraic closure, we may

abstractly denote a system by,

$$y = Su \tag{3.1}$$

where S is the mapping (operator) representing the system. The system, S, is primarily determined by two factors: the component types, and the ways the components are interconnected. In the case of an LSC, a system consists of a fixed interconnection of several Linear Sequential Circuits, each in turn which is a fixed interconnection of components or devices. Components can be scalars, adders, delayors or linear gates. The form of the algebraic connection model, known as the Component Connection Model, can be conceptually deduced from Figure 3.1.

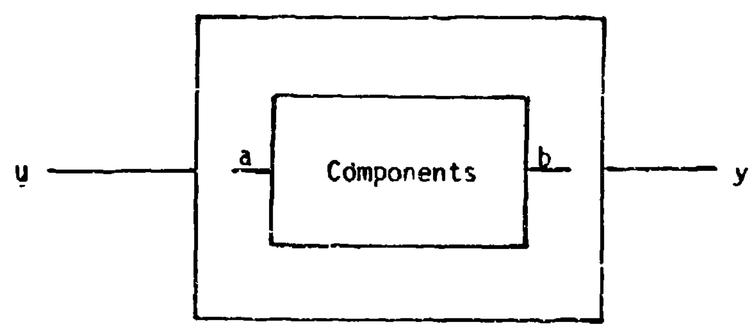

Figure 3.1: System Representation

In Figure 3.1, u and y represent overall system inputs and outputs (finite field valued sequences) while a and b are, respectively, component input and output sequences over the same field.

By a "Component Connection Model" is meant a system model wherein the components and connections are characterized by separate equations. In particular, one assumes that each component in a system is characterized by the D-transform equation,

$$b_i(D) = [(g_i)f_i(D)]\ a_i(D)\ , \qquad i = 1, 2, \ldots n \tag{3.2}$$

where $a_i(D)$ and $b_i(D)$ are respectively the D-transforms of the input and output sequences of the i^{th} component. The dynamics of the i^{th} component are represented by $f_i(D)$ and the scalar constant, g_i, is the gain of the i^{th} component. For the purpose of fault analysis, it is assumed that the component dynamics, $f_i(D)$, do not change even after a fault has occurred and all failures manifest themselves as changes in g_i with memory elements and connections good. This guarantees that a linear system fails linearly. Such a fault model includes the usual "open" and "short circuit" faults and "stuck-on-zero" faults. "Stuck-on-one" faults can not be included in a strictly linear theory since a device which is "stuck-on-one" is nonlinear. "Stuck-on-one" faults are, however, included in the generalization of the theory to Affine Sequential Circuits described in Chapter V.

In actual practice one normally works with the n separate component equations like (3.2). Notationally this may be combined into the single matrix equation,

$$b(D) = [(G)F(D)]\ a(D) \tag{3.3}$$

where

$$b(D) = \text{Col.}\ (b_i(D))$$

$$a(D) = \text{Col.}\ (a_i(D))$$

$$G = \text{diag.}\ (G_i)$$

$$F(D) = \text{diag.}\ (f_i(D))$$

To obtain a mathematical model for the connections, redraw Fig. 3.1 as in Fig. 3.2 where the components and connections are shown separately. The connections may be viewed as a separate multiple input - multiple output component.

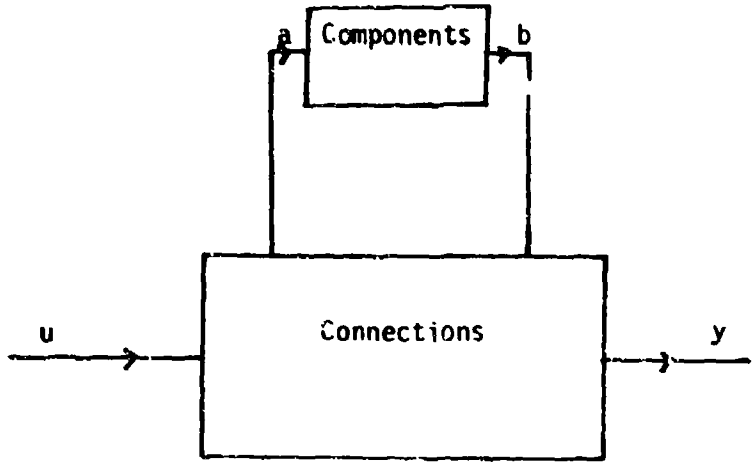

Figure 3.2: Component Connection Model Representation

Since connection elements are linear and algebraic, and connections can be characterized by linear algebraic constraints (Adders, scalers, etc.) the connection model can be represented by the matrix equation,

$$\begin{bmatrix} a(D) \\ \hline y(D) \end{bmatrix} = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right] \begin{bmatrix} b(D) \\ \hline u(D) \end{bmatrix} \tag{3.4}$$

Entries in the L matrices take their values in the same finite field over which the LSC under study is defined. For digital circuits the L_{ij} matrices are usually permutation matrices describing how the outputs of one component are connected to the inputs of another. In Equation (3.4), u(D) and y(D) are respectively the D-transforms of the externally accessible inputs and outputs of the LSC. The digital version of the component connection model is sufficiently general to include most LSC's, although it is not universal. Equation (3.4) has the symbolic representation shown in Figure 3.3.

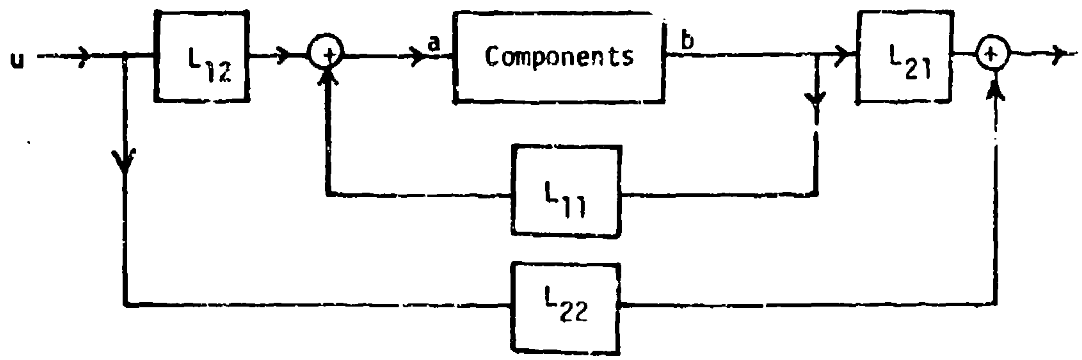

Figure 3.3: Symbolic Representation of L_{ij} Matrices

In the following example, L_{ij}, G and F(D) matrices are determined for the LSC shown in Fig. 3.4.

Example 3.1:

Consider the system defined over GF(2) as shown in Figure 3.4.

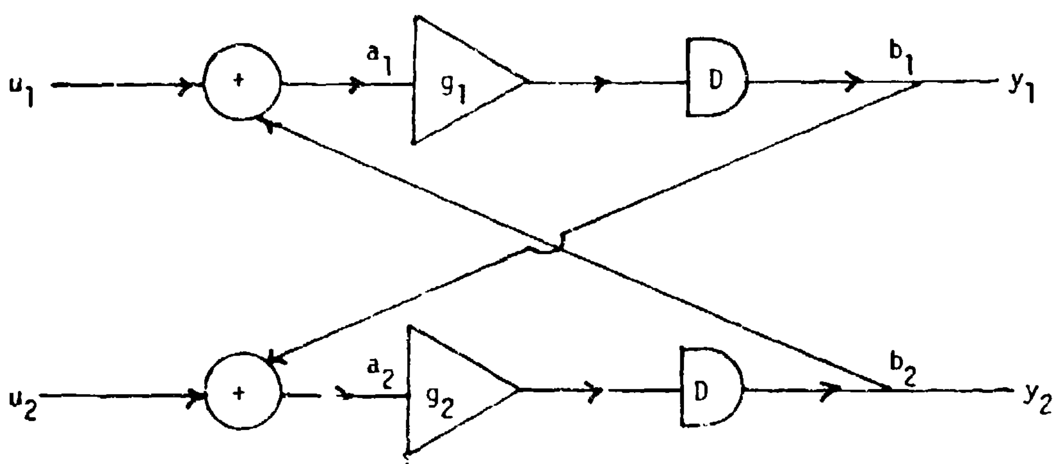

Figure 3.4: Figure for Example 3.1

The L matrices for the system described in Figure 3.5 are given by

$$\begin{bmatrix} a_1 \\ a_2 \\ \hline y_1 \\ y_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{bmatrix} b_1 \\ b_2 \\ \hline u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} g_{1/D} & 0 \\ 0 & g_{2/D} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \begin{bmatrix} 1/D & 0 \\ 0 & 1/D \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

Simultaneous solution of equations (3.1), (3.3) and (3.4) yields a complete description of the LSC input-output relationship of the form,

$$y(D) = [S(D)]\, u(D) \tag{3.5}$$

$$= [L_{22} + L_{21}(1-(G)F(D)L_{11})^{-1}(G)F(D)L_{12}]u(D) \tag{3.6}$$

That is, the overall system operator is given in terms of the component dynamics, $F(D)$, component parameters, G, and connections, L_{ij}, by the equality,

$$S(D) = L_{22} + L_{21}(1-(G)F(D)L_{11})^{-1}(G)F(D)L_{12} \tag{3.7}$$

$S(D)$ is a matrix of rational functions in D with coefficients in a finite field, whose poles and zeros are well defined elements of the algebraic closure of the finite field.

In view of the hypothesis that all LSC failures manifest themselves as changes in G, with $F(D)$ and L_{ij} remaining constant, it is natural to view Equation (3.7) as,

$$S(D) = f_D(G) \tag{3.8}$$

where f_D is a nonlinear function even though S is a linear operator. The function, f_D, is entirely determined by the component dynamics and the connections relating component parameter values to the system's input and output. The function, f_D, is called the connection function.[5] Equation (3.8) is in just the right form for the study of fault analysis because faults are assumed to manifest themselves only as changes in G.

3.2 Fault Analysis:

The following theorem gives the relationship between S(e) which may be computed from the zero state impulse response of the LSC and S(D).

Theorem 3:

Given an LSC described by $y(D) = S(D)u(D)$. Let $\{u_k\} = \{e^{+k}\}$, $k = 0, \pm 1, \pm 2, \ldots$, where e is an element in the algebraic closure of a finite field over which the LSC is defined. Then $S(e) \mid D = e$ (3.9)

<u>Proof</u>:

Consider, first, the single input - single output case. In this case S(D) is a rational function in D with well defined poles and zeros that can be expressed as

$$S(D) = \frac{a_0 + a_1 D + a_2 D^2 + \text{---} + \text{---} + a_n D^n}{1 + b_1 D + b_2 D^2 + \text{---} + \text{---} + b_m D^m} \tag{3.10}$$

which implies

$$y(D) = \frac{a_0 + a_1 D + a_2 D^2 + \text{------} + a_n D^n}{1 + b_1 D + b_2 D^2 + \text{------} + b_m D^m} u(D)$$

Equivalently,

$$y(D)[1 + b_1 D + b_2 D^2 + \cdots + b_m D^m] = [a_o + a_1 D + a_2 D^2 + \cdots + a_n D^n]u(D). \tag{3.10A}$$

Since $D^m y(D) = \{y_{k+m}\}$ and $D^n u(D) = \{u_{k+m}\}$, (3.10A) becomes,

$$\{y_k\} + b_1\{y_{k+1}\} + \cdots + b_m\{y_{k+m}\} = a_o\{u_k\} + a_1\{u_{k+1}\} + \cdots + a_n\{u_{k+n}\} \tag{3.11}$$

Let $u_k = \{e^k\}$, $\quad k = 0, \pm 1, \pm 2, \ldots .$

Then $y_k = \{ye^{+k}\}$ (3.12)

Substituting (3.12) into (3.11) yield,

$$\{ye^k\}[1 + b_1 e + b_2 e^2 + \cdots + b_m e^m] = \{e^k\}[a_o + a_1 e + \cdots + a_n e^n]$$

or

$$\{y_k\} = \frac{a_o + a_1 e + a_2 e^2 + \cdots + a_n e^n}{1 + b_1 e + b_2 e^2 + \cdots + b_m e^m} \{u_k\}$$

$$= S(e)\{u_k\}, \text{ where } S(e) = S(D) \mid D = e \tag{3.13}$$

The multiple input - multiple output result follows easily by applying the above analysis over all possible input-output pairs with all inputs zero except the one corresponding to the pair in question.

Given the Theorem 3, Equation (3.6) may now be written as

$$S(e) = f_e(G) = [L_{22} + L_{21}(1-(G)F(e)L_{11})^{-1}(G)F(e)L_{12}]. \tag{3.14}$$

If (3.14) can be solved for G, the fault analysis of the Linear Sequential Circuit is complete. Unfortunately, however, most LSC's have more components than input-output equation, in which case (3.14) represents a set of equations that has more unknowns than the number of equations. However, this difficulty can be overcome by exploiting the LSC dynamics. Indeed, this exploitation of the circuit dynamics is the key to the spectral theoretic approach because it permits the

number of externally accessible LSC terminals required for fault analysis of an LSC to be reduced from that required for a combinatorial circuit of similar complexity rather than increased as is the case with traditional fault analysis techniques. To see this, note that Equation (3.14) is valid for any e in the algebraic closure of a finite field over which the LSC is defined and that the resulting equations are dependent on the choice of e. As such, more equations can be created without changing the number of unknowns by writing the set of simultaneous equations for different e's in the algebraic closure as,

$$\begin{aligned} S(e_1) &= f_{e_1}(G) = [L_{22} + L_{21}(1-(G)F(e_1)L_{11})^{-1}(G)F(e_1)L_{12}] \\ S(e_2) &= f_{e_2}(G) = [L_{22} + L_{21}(1-(G)F(e_2)L_{11})^{-1}(G)F(e_2)L_{12}] \\ &\vdots \\ S(e_k) &= f_{e_k}(G) = [L_{22} + L_{21}(1-(G)F(e_k)L_{11})^{-1}(G)F(e_k)L_{12}]. \end{aligned} \tag{3.15}$$

In the case of a combinatorial circuit f_{e_i} is independent of e_i hence the additional equations are not independent and do not simplify the fault analysis process. In the case of a Linear Sequential Circuit, Equations (3.15) will be independent and solvable for G even though a single Equation (3.14) is not solvable.

With the above theory, one can bormulate the fault analysis algorithm. The procedure consists of the following steps.

(i) Measure the zero state impulse response of an LSC under study.

(ii) Compute $S(e_i)$ from (2.25) for various elements $e_1, e_2, e_3, \ldots, e_k$ in the extension of the finite field over which the LSC under study is defined. The number of e's that one should choose depends on how many component parameters are to be solved.

(iii) From $S(e_1)$, $S(e_2)$,...., $S(e_k)$ calculated in Step (ii) and from the known matrices L_{ij}, write the simultaneous equations (3.15).

(iv) Solve the equations (3.15) for G.

In the next section, the algorithm for solving (3.15) is formulated.

3.3 Equation Solving:

A procedure is now given for solving the set of simultaneous equations (3.15). This procedure is based on a "term expansion" algorithm developed by Ransom[14] for analog fault analysis.

Let $T(D) = ((G)F(D))^{-1}$. Equation (3.7) becomes,

$$S(D) = L_{22} + L_{21}(T(D)-L_{11})^{-1}L_{12}, \quad i = 1,2,\ldots,k \tag{3.16}$$

Next, perform the "term expansion" of the "inverse" in (3.16).

$(T(D)-L_{11})^{-1} = \frac{1}{\Delta}\,\mathrm{adj.}(T(D)-L_{11})^t$ where t denotes transposition, $\Delta =$ $(T(D)-L_{11})$, and adj.$(T(D)-L_{11})$ denotes the matrix whose i_1j^{th} element is the i_1j^{th} cofactor of $(T(D)-L_{11})$.

Taking the "vec" operation on both sides of (3.16) yields

$$\mathrm{vec}(S) = \mathrm{vec}(L_{22}) + (L_{12}{}^t \otimes L_{21}) \frac{1}{V_a\rho_a} G_a\rho_a \tag{3.17}$$

where $\otimes$ denotes the Kronecker matrix product.[19]

In arriving at (3.17) one needs to use the formula vec(ABC) = $(C^T \otimes A)\mathrm{vec}(B)$, the equalities $\Delta = V_a\rho_a$ and vec adj.$(T-L_{11})^t = G_a\rho_a$ where V_a is a row vector, G_a is a matrix and

$$\rho_a = \begin{bmatrix} 1 \\ T^{C_1} \\ \vdots \\ T^{C_n} \end{bmatrix}$$

Here n is the dimension of $T(D)$ and T^{C_i} is a vector whose elements are all product combinations of the diagonal elements of T taken i at a time. That is, for an LSC with three g's, $\rho_a = [1, g_1, g_2, g_3, g_1g_2, g_1g_3, g_2g_3, g_1g_2g_3]$. Here the elements of V_a and G_a are constants in an extension of a finite field. An algorithm for determining V_a and G_a is given in reference and will not be repeated.

One can write,

$$V_a\rho_a = V\rho + \Gamma$$

$$G_a\rho_a = G\rho + \lambda \tag{3.18}$$

where ρ, V, G are ρ_a, V_a and G_a, respectively, with their first element (column) deleted. The first element (column) of V_a and G_a are respectively, Γ and λ.

Substituting (3.18) into (3.17) yields,

$$\{(L_{12}^t \otimes L_{21})G - \mathrm{vec}(T-L_{22})V\}\rho = -(L_{12}^t \otimes L_{21})\lambda - \mathrm{vec}(T-L_{22})\Gamma$$

or,

$$\psi\rho = \delta \tag{3.19}$$

Writing Equation (3.19) for various values of e_i, one can obtain the desired form of set of simultaneous equations,

$$\begin{aligned} \psi(e_1)\rho &= \delta(e_1) \\ \psi(e_2)\rho &= \delta(e_2) \\ &\vdots \\ \psi(e_k)\rho &= \delta(e_k) \end{aligned}$$

which can be written more compactly with the obvious notational definitions as,

$$\tilde{\psi}\rho = \tilde{\delta} \tag{3.20}$$

In general, $\tilde{\psi}$ may have linearly dependent columns (over an extension field containing all of the above e_i's) i.e., $a_ic_i + a_jc_j +$

$\alpha_k c_k + \cdots = c_m$.

Where $c_i, c_j, \ldots, c_m$ are columns of $\tilde{\psi}$, and $\alpha_i, \alpha_j, \ldots$ are scalers which may be elements of the extension field. Thus if c_m is the dependent column, delete c_m from $\tilde{\psi}$ and R_m from $\tilde{\rho}$. Add $\alpha_i R_m$ to row R_i, $\alpha_j R_m$ to row R_j, etc.. Repeat this process until all linear dependencies among the columns of $\tilde{\psi}$ have been deleted. Denote the resulting equations by

$$\hat{\psi}\,\hat{\rho} = \tilde{\delta} \tag{3.21}$$

Here $\tilde{\delta}$ is a vector of the extension field elements, $\hat{\psi}$ has a left inverse given by,

$$\hat{\psi}^{-L} = (\hat{\psi}^t\,\hat{\psi})^{-1}\,\hat{\psi}^t, \tag{3.22}$$

Hence, one can solve for $\hat{\rho}$ as,

$$\hat{\rho} = \hat{\psi}^{-L}\,\tilde{\delta}, \tag{3.23}$$

which is a vector of elements in the extension field. Finally, one desires to compute the g_i's from $\hat{\rho}$. One can express $\hat{\rho}$ as $\hat{\rho} = \beta\rho$ where β is a known matrix of extension field elements. There is also an additional constraint that the g_i's lie in GF(p), even though β and $\hat{\rho}$ are composed of extension field elements. Since every extension field element can be uniquely represented as a m^{th} order polynomial in an indeterminant α with coefficients in GF(p),[13] Equation (3.23) can be written

$$\sum_{\gamma=0}^{m} [\beta_\gamma \rho]\alpha^\gamma = \sum_{r=0}^{m} [q_r]\alpha^r$$

where β_γ and q_r are the coefficients of the polynomial.

Upon equating coefficients of like powers of α, one obtains a set of simultaneous equations in GF(p) that are to be solved for the g_i's.

These equations can be solved by any standard solution techniques in GF(p). In particular, in GF(2), one can set up a Boolean expression, which after simplification, reduces to a list of all possible sets of component values consistent with the specified data.

In the following chapter, several examples of LSC fault analysis are considered in which the technique described above is used for solving the set of simultaneous equations arising in the fault analysis.

CHAPTER IV

FAMILY OF EXAMPLES

Example 4.1: Consider the LSC shown in Figure 4.1, defined over GF(2).

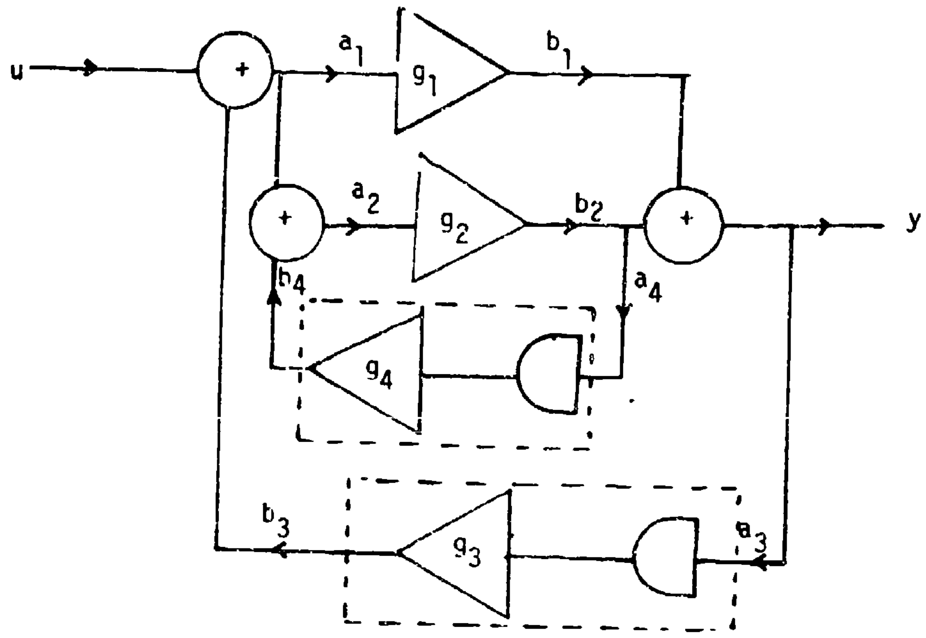

Figure 4.1: A LSC for Example 4.1

Assume that the zero state impulse response for the LSC in Figure 4.1 is measured and is given by, $h_0 = 1$, $h_1 = 1$, $h_2 = 1, \ldots\ldots$

It is desired to compute all possible values of the gains g_1, g_2, g_3, g_4 which are compatible with the zero state impulse response data given above.

Solution: The connection equations for this circuit are as follows.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ u \end{bmatrix} \tag{4.1}$$

and,

$$[(G)F(D)] = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3/D & 0 \\ 0 & 0 & 0 & g_4/D \end{bmatrix}$$

$$\begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/D & 0 \\ 0 & 0 & 0 & 1/D \end{bmatrix} \tag{4.2}$$

Substituting the matrices L_{ij}, G and F(D) obtained from (4.1) and (4.2) into (3.6) and then using the term expansion algorithm, one obtains,

$$\begin{bmatrix} 1 & 1 & \frac{S(D)}{D} & \frac{S(D)}{D} & \frac{S(D)}{D} & \frac{1}{D} & \frac{S(D)}{D^2} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_2 g_3 \\ g_2 g_4 \\ g_1 g_3 \\ g_1 g_2 g_4 \\ g_1 g_2 g_3 g_4 \end{bmatrix} \begin{bmatrix} S(D) \end{bmatrix} \tag{4.3}$$

Equation (4.3) is of the form,

$$\Psi \rho = \delta$$

where zero entries in Ψ and in the corresponding rows of ρ are deleted. The solution of the set of Equation (4.3) is carried out in $(GF(2^3))$. Choose the elements $e_1=\alpha$, $e_2=\alpha^2$, $e_3=\alpha^3$, $e_4=\alpha^4$, $e_5=\alpha^5$, $e_6=\alpha^6$ from $GF(2^3)$ where α is an indeterminant. Use (2.25) to compute $S(e_i)$ for $i = 1, 2, 3, 4, 5, 6$ from the zero state impulse response data. They are given as in (4.4)

$$\begin{aligned} S(e_1) &= \alpha^5 \\ S(e_2) &= \alpha^3 \\ S(e_3) &= \alpha^2 \\ S(e_4) &= \alpha^6 \\ S(e_5) &= \alpha \\ S(e_6) &= \alpha^4 \end{aligned} \tag{4.4}$$

The set of simultaneous equations (4.3) are,

$$\begin{bmatrix} 1 & 1 & \frac{S(e_1)}{e_1} & \frac{S(e_1)}{e_1} & \frac{S(e_1)}{e_1} & \frac{1}{e_1} & \frac{S(e_1)}{e_1^2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \frac{S(e_6)}{e_6} & \frac{S(e_6)}{e_6} & \frac{S(e_6)}{e_6} & \frac{1}{e_6} & \frac{S(e_6)}{e_6^2} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_2g_3 \\ g_2g_4 \\ g_1g_3 \\ g_1g_2g_4 \\ g_1g_2g_3g_4 \end{bmatrix} = \begin{bmatrix} S(e_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ S(e_6) \end{bmatrix} \tag{4.5}$$

Substituting (4.4) into (4.5),

$$\begin{bmatrix} 1 & 1 & \alpha^4 & \alpha^4 & \alpha^4 & \alpha^6 & \alpha^3 \\ 1 & 1 & \alpha & \alpha & \alpha & \alpha^5 & \alpha^6 \\ 1 & 1 & \alpha^6 & \alpha^6 & \alpha^6 & \alpha^4 & \alpha^3 \\ 1 & 1 & \alpha^2 & \alpha^2 & \alpha^2 & \alpha^3 & \alpha^5 \\ 1 & 1 & \alpha^3 & \alpha^3 & \alpha^3 & \alpha^2 & \alpha^5 \\ 1 & 1 & \alpha^5 & \alpha^5 & \alpha^5 & \alpha & \alpha^6 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_2g_3 \\ g_2g_4 \\ g_1g_3 \\ g_1g_2g_4 \\ g_1g_2g_3g_4 \end{bmatrix} = \begin{bmatrix} \alpha^5 \\ \alpha^3 \\ \alpha^2 \\ \alpha^6 \\ \alpha \\ \alpha^4 \end{bmatrix} \tag{4.6}$$

(4.6) is of the form,

$$\tilde{\Psi}\ \rho = \tilde{\delta}$$

In (4.6) the matrix, $\tilde{\Psi}$, has only three linearly independent columns and can be reduced to $\hat{\Psi}$ by using the equalities,

$$c_1 = c_2$$

$$c_3 = c_4 = c_5$$

$$c_3 + c_6 = c_7$$

where c_i is the i^{th} column of $\tilde{\Psi}$. The resulting equations are,

$$\begin{bmatrix} 1 & \alpha^4 & \alpha^6 \\ 1 & \alpha & \alpha^5 \\ 1 & \alpha^6 & \alpha^4 \\ 1 & \alpha^2 & \alpha^3 \\ 1 & \alpha^3 & \alpha^2 \\ 1 & \alpha^5 & \alpha \end{bmatrix} \begin{bmatrix} g_1 + g_2 \\ g_2g_3 + g_2g_4 + g_1g_3 + g_1g_2g_3g_4 \\ g_1g_2g_4 + g_1g_2g_3g_4 \end{bmatrix} = \begin{bmatrix} \alpha^5 \\ \alpha^3 \\ \alpha^2 \\ \alpha^6 \\ \alpha \\ \alpha^4 \end{bmatrix} \tag{4.7}$$

The equation (4.7) is of the form,

$$\hat{\psi} \quad \hat{\rho} = \delta$$

where,

$$\hat{\rho} = \begin{bmatrix} g_1 + g_2 \\ g_2g_3 + g_2g_4 + g_1g_3 + g_1g_2g_3g_4 \\ g_1g_2g_4 + g_1g_2g_3g_4 \end{bmatrix}$$

or,

$$\hat{\rho} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_2g_3 \\ g_2g_4 \\ g_1g_3 \\ g_1g_2g_4 \\ g_1g_2g_3g_4 \end{bmatrix}$$

Now multiplying both sides of (4.7) by ψ^{-L}, one obtains,

$$\hat{\rho} = \begin{bmatrix} g_1 + g_2 \\ g_2g_3 + g_2g_4 + g_1g_3 + g_1g_2g_3g_4 \\ g_1g_2g_4 + g_1g_2g_3g_4 \end{bmatrix} = \psi^{-L} \delta \quad = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad (4.8)$$

Note that all coefficients are in GF(2), so the g_i's can be solved for directly by expanding the Boolean equation, where "+" in GF(2) is interpreted as an "EXCLUSIVE OR" operation in Boolean Algebra.

$$(g_1+g_2)\ (g_2g_3+g_2g_4+g_1g_3+g_1g_2g_3g_4)\ \overline{(g_1g_2g_4+g_1g_2g_3g_4)} = 1 \qquad (4.9)$$

Expanding (4.9) into sum of products,

$$\bar{g}_1 g_2 g_3 g_4 \vee g_1 \bar{g}_2 g_3 g_4 \vee g_1 \bar{g}_2 g_3 \bar{g}_4 = 1 \qquad (4.10)$$

which yields three possible solutions consistent with the measured impulse response. They are,

$$g_1 = 0,\ g_2 = 1,\ g_3 = 1,\ g_4 = 1.$$
$$g_1 = 1,\ g_2 = 0,\ g_3 = 1,\ g_4 = 1.$$
$$g_1 = 1,\ g_2 = 0,\ g_3 = 1,\ g_4 = 0. \qquad (4.11)$$

Example 4.2 : Consider the LSC shown in figure 4.2, defined over GF(2). This circuit differs from that given in figure 4.1 in that it has two delays in series with component 2.

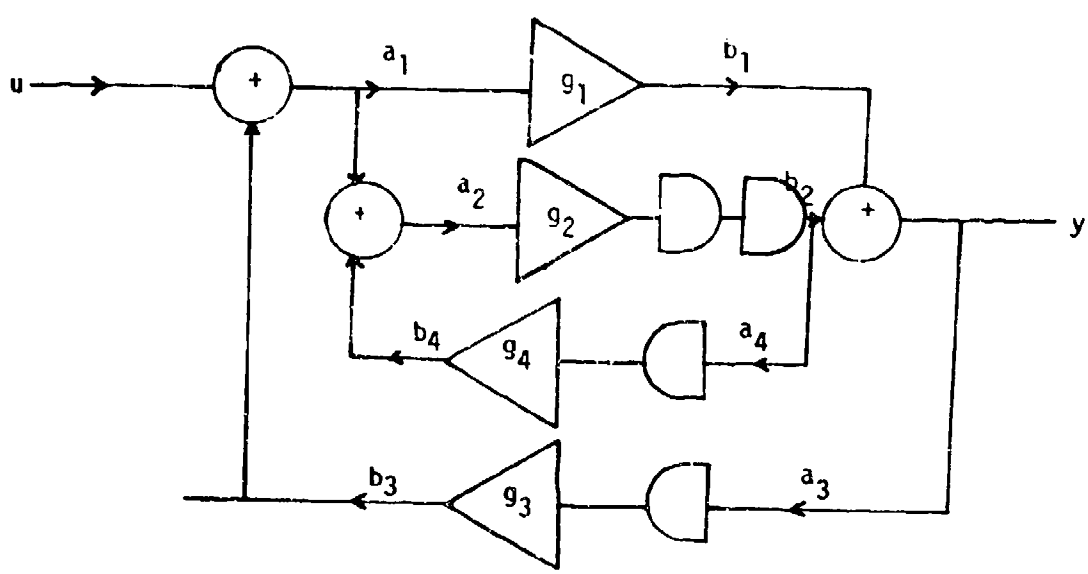

Figure 4.2: A LSC for Example 4.2

Assume that the measured zero state impulse response is

$\{ h \} = \{ 1,1,0,1,0,0,1,1,.......... \}$

Solution : L_{ij} matrices are same as given in (4.1) and

$$[(G(D)] = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ 0 & g_2 & 0 & 0 \\ 0 & 0 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/D^2 & 0 & 0 \\ 0 & 0 & 1/D & 0 \\ 0 & 0 & 0 & 1/D \end{bmatrix} \quad (4.12)$$

Substituting the matrices L_{ij}, G and F(D) obtained from (4.1), (4.12) into (3.6) and then using the term expansion algorithm , one obtains the set of equations,

$$\begin{bmatrix} 1 & \frac{1}{D} & \frac{S(D)}{D^3} & \frac{S(D)}{D} & \frac{1}{D^3} & \frac{S(D)}{D^4} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_2 g_3 \\ g_2 g_4 \\ g_1 g_3 \\ g_1 g_2 g_4 \\ g_1 g_2 g_3 g_4 \end{bmatrix} = [S(D)] \quad (4.13)$$

(4.13) is of the form,

$$\Psi \ \rho = \delta$$

where zero entries in Ψ and in corresponding rows of ρ are deleted. Attempting to use the e's from $GF(2^3)$, one obtains,

$$S(e_1) = S(\alpha) = \alpha^3$$
$$S(e_2) = S(\alpha^2) = \alpha^6$$
$$S(e_3) = S(\alpha^3) = \alpha^5/0 \quad \text{i.e., undefined}$$
$$S(e_4) = S(\alpha^4) = \alpha^5$$
$$S(e_5) = S(\alpha^5) = \alpha^3/0 \quad \text{i.e., undefined}$$
$$S(e_6) = S(\alpha^6) = \alpha^6/0 \quad \text{i.e., undefined}$$

Note that $e_3 = \alpha^3$, $e_5 = \alpha^5$, $e_6 = \alpha^6$ are poles of $S(e)$ and hence these three elements can not be used to set up the set of simultaneous equations (2.26). As such one can write three fault analysis equations in $GF(2^3)$ which are not sufficient to solve for the g_i's. However, $S(e_i)$ is defined in $GF(2^4)$. Tables of multiplication and addition for $GF(2^4)$ are given in reference [15].

Equation (2.25) can now be used to compute $S(e_i)$ from the zero state impulse response data. These are given by,

$$S(e_1) = S(\alpha) = \alpha^{11}$$
$$S(e_2) = S(\alpha^2) = \alpha^7$$
$$S(e_3) = S(\alpha^3) = \alpha^6$$
$$S(e_4) = S(\alpha^4) = \alpha^{14}$$
$$S(e_5) = S(\alpha^5) = 1$$
$$S(e_6) = S(\alpha^6) = \alpha^{12}$$
$$S(e_7) = S(\alpha^7) = \alpha^8 \tag{4.14}$$

Upon expanding (4.13) for each e_i from $GF(2^4)$ and substituting in (4.14) and in (4.13), the set of simultaneous equations obtained are,

$$\begin{bmatrix} 1 & \alpha^{13} & \alpha^{8} & \alpha^{8} & \alpha^{10} & \alpha^{12} & \alpha^{7} \\ 1 & \alpha^{11} & \alpha & \alpha & \alpha^{5} & \alpha^{9} & \alpha^{14} \\ 1 & \alpha^{9} & \alpha^{12} & \alpha^{12} & \alpha^{3} & \alpha^{6} & \alpha^{9} \\ 1 & \alpha^{7} & \alpha^{2} & \alpha^{2} & \alpha^{10} & \alpha^{3} & \alpha^{13} \\ 1 & \alpha^{5} & 1 & 1 & \alpha^{10} & 1 & \alpha^{10} \\ 1 & \alpha^{3} & \alpha^{9} & \alpha^{9} & \alpha^{6} & \alpha^{12} & \alpha^{3} \\ 1 & \alpha & \alpha^{2} & \alpha^{2} & \alpha & \alpha^{9} & \alpha^{10} \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_2g_3 \\ g_2g_4 \\ g_1g_3 \\ g_1g_2g_4 \\ g_1g_2g_3g_4 \end{bmatrix} \begin{bmatrix} \alpha^{11} \\ \alpha^{7} \\ \alpha^{6} \\ \alpha^{14} \\ 1 \\ \alpha^{12} \\ \alpha^{8} \end{bmatrix} \tag{4.15}$$

Equation (4.15) is of the form,

$$\tilde{\psi}\,\rho = \tilde{\delta}$$

After deleting linearly dependent columns of $\tilde{\psi}$ and then using the technique described in Example 4.1, one obtains,

$$\hat{\rho} = \begin{bmatrix} \alpha^{9}g_1 + \alpha^{8}g_1g_2g_3g_4 \\ g_1 + \alpha g_2 \\ \alpha^{7}g_1g_2g_4 + \alpha^{2}g_1g_2g_3g_4 \\ \alpha^{10}g_1 + \alpha^{11}g_1g_3 + \alpha^{7}g_1g_2g_3g_4 \end{bmatrix} = \begin{bmatrix} \alpha^{9} \\ 1 + \alpha \\ 0 \\ \alpha^{14} \end{bmatrix} \tag{4.16}$$

Equation (4.16) yields a unique solution,

$$g_1 = 1,\ g_2 = 1,\ g_3 = 1,\ g_4 = 0\ . \tag{4.17}$$

One can check this solution by finding an impulse response of the LSC in Figure 4.2 by using the values of gain in (4.15).

The important point here is that with the additional delays, the internal component parameters can be determined exactly from the impulse response whereas without the delay the best that one can do is to obtain a list of three possible faults.

CHAPTER V

FAULT ANALYSIS - AFFINE SEQUENTIAL CIRCUITS

5.1 Introduction:

Mathematically, a "two-sided" Affine Sequential Circuit (ASC) over a finite field is represented by a set of difference equations,

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Ju_k + w_k, \quad -\infty < k < \infty \\ x_0 &= \underline{x}, \end{aligned} \tag{5.1}$$

where $\{w_k\}$ is a sequence of constants over a finite field and w_k is some constant at time k. Other terms in (5.1) are as defined in Chapter II.

In cases where ASC's are not given in the form shown in (5.1), a change of variables and some manipulations will yield the desired form (5.1). Hence, without loss of generality, (5.1) can be considered as a standard form for the purpose of fault analysis in Affine Sequential Circuits.

For Affine Sequential Circuits the present value of the output depends not only on the present input value and the present state (as is the case with LSC's) but also on some constants defined over a finite field. Since the mathematical representation of ASC's and LSC's only differ by some constant value in the output equation, the existence and uniqueness theory for the solution of (5.1) is similar to that discussed in Chapter II for LSC's and will not be repeated here.

5.2 Component Connection Model:

Since Affine Sequential Circuits have affine components, each component of an ASC is characterized by the D-transfer equation,

$$b_i(D) = [(g_i)f_i(D)]\ a_i(D) + [q_i(D)],\ i = 1,2,\ldots,n \qquad (5.2)$$

where, $a_i(D)$ is the D-transform of the input sequence to the i^{th} component,

$b_i(D)$ is the D-transform of the ouput sequence to the i^{th} component,

$f_i(D)$ represents the dynamics of the i^{th} component,

g_i, a scalar of the underlying finite field, is the gain of the linear part of i^{th} component.

Let n_i be a bias sequence added to the i^{th} component. The D-transform of this bias sequence, $n_i(D)$ is given by,

$$n_i(D) = \left(\frac{D}{D-1} \right) n_i$$

$q_i(D)$ is, then, a product of $n_i(D)$ and whatever part of the component dynamics the bias signal passes through.

For the purpose of fault analysis, it is assumed that the component dynamics $f_i(D)$ remain constant and all faults manifest themselves as changes in g_i and n_i. Such a fault model includes "stuck-on-one" faults which are not included in the case of LSC's.

One may note that an ASC over GF(2) can be viewed in either of two ways; first as an LSC in which a constant bias source is introduced and secondly as an LSC into which NOT gate has been inserted. These two viewpoints are equivalent since one can construct a NOT gate with a bias adder and conversely.

For notational simplicity, the n scalar equations, (5.4), can be combined into a single matrix equation,

$$b(D) = [(G)F(D)]\ a(D) + [Q(D)] \qquad (5.5)$$

where $\quad b(D) = \text{column}(b_i(D)),$

$$a(D) = \text{column}(a_i(D)),$$

$$G = \text{diag.}(g_i),$$

$$F(D) = \text{diag.}(f_i(D)),$$

$$Q(D) = \text{diag.}(q_i(D)).$$

Example 5.1:

This example illustrates the formulation of Equation (5.5) for the component shown in Figure 5.1.

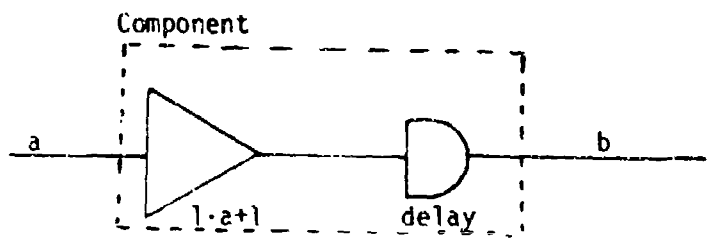

Figure 5.1: Figure for Example 5.1

Let $\quad G = 1$

$\quad [illegible] = 1$

Then

$$b(D) = \left[\frac{1}{D}\right]a(D) + \left(\frac{D}{D-1}\right)\left(\frac{1}{D}\right) 1.$$

As in the previous discussion of the component connection model, the connection structure of an Affine Sequential Circuit is described by the algebraic constraints,

$$\begin{bmatrix} a(D) \\ y(D) \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} b(D) \\ u(D) \end{bmatrix} \tag{5.6}$$

Simultaneous solution of (5.5) and (5.6) yields,

$$y(D) = [\, L_{22} + L_{21}(I - (G)F(D)\, L_{11})^{-1}(G)F(D)L_{12} \,]\, u(D)$$

$$+ [\, L_{21}(I - (G)F(D)\, L_{11})^{-1}\, (Q(D)) \,] \tag{5.7}$$

Denote $\quad S(D) = [L_{22} + L_{21}(I - (G)F(D)L_{11})^{-1}(G)F(D)L_{12}]$ and

$$T(D) = [L_{21}(I - (G)F(D)L_{11})^{-1}(Q(D))] \tag{5.8}$$

Then (5.7) becomes,

$$y(D) = S(D)\, u(D) + T(D) \tag{5.9}$$

From (5.8) it is seen that the D-transform of a transfer function, $S(D)$, is the linear part of the ASC.

5.3 Fault Analysis:

Fault analysis in Affine Sequential Circuits involves developing techniques for determining G and η. The unknown, G, can be solved for by considering the linear part of an Affine Sequential Circuit.

If one can determine the impulse response of the linear part of an ASC, Equation (3.15) may be set up for several e's in the algebraic closure of a finite field and then can solve for the unknown g_i's using the algorithm described in Chapter III. The impulse response of the linear part of an ASC is determined as follows.

Observe the response of the ASC due to an input sequence $\{u_k\} = \{0\}$. When this response becomes periodic, apply an impulse sequence to the ASC under study. The impulse response of the linear part of an ASC is then given by the difference of two responses i.e., response obtained after and before the impulse sequence is applied.

After the g_i's are determined, it remains to determine the η_i's.

Taking the D-transform of Equation (5.1) and then comparing with (5.9), one obtains,

$$W(D) = T(D) \;. \tag{5.10}$$

where W(D) is the D-transform of the sequence $\{w_k\}$.

If $\{w_k\}$ can be determined from the measurement on an ASC, n can be determined from (5.10). The sequence $\{w_k\}$ is determined from the measurements on an ASC as follows:

Let an input sequence $\{u_k\}$ be an impulse sequence, then is follows from (5.1) that

$$h_k' = CA^k\underline{x} + CA^{k-1}B + w_k, \quad k > 0 \tag{5.10A}$$

where h_k' is the impulse response of an Affine Sequential Circuit. Again using the controllability criteria as discussed in Chapter III, one can write,

$$CA^k\underline{x} = \sum_{j=1}^{\tilde{k}_0} CA^kB\,\tilde{b}_{-j} \tag{5.11}$$

where $\tilde{b}_{-j}$, $\tilde{k}_0 < -j < 0$ is a sequence of inputs that drives the minimal realization of an ASC from the zero state at time $k = \tilde{k}_0$ to $\underline{x}$ at time $k = 0$. Such a sequence exists for a minimal realization if $\tilde{k}_0 < -\tilde{\delta}$, where $\tilde{\delta}$ is the dimension of the minimal realization. One has the equality,

$$[CA^{k-1}B] = h_k \tag{5.12}$$

where h_k is the impulse response (at time k) of the linear part of an ASC.

Substituting (5.11) and (5.12) into (5.10) yields,

$$h_k' = \sum_{j=1}^{\tilde{k}_0} CA^kB\,\tilde{b}_{-j} + h_k + w_k, \quad k > 0. \tag{5.13}$$

From (5.13) one can solve for w_k (and $\tilde{b}_{-j}$). No attempt is made here to develop any technique for solving w_k from (5.13). Instead an

assumption is made that $x_0 = \underline{x} = 0$ which simplifies the expression (5.10), yielding

$$h'_k = h_k + w_k$$

or

$$w_k = h'_k - h_k \tag{5.14}$$

With the above theory, one can formulate the fault analysis algorithm on ASC's. The procedure consists of the following steps:

(i) Solve for g_i, from the impulse response of the linear part of an ASC and then use steps (i) through step (iv) given in Chapter III for the purpose of fault analysis on LSC's;

(ii) Substitute the value of g_i's (obtained from step (i)) in T(D);

(iii) Measure the impulse response of the Affine Sequential Circuit and using (5.14) obtain $\{w_k\}$;

(iv) Obtain the D-transform of the sequence $\{w_k\}$ i.e., obtain W(D);

(v) Using equality (5.10), solve for η, since η is contained in T(D).

5.4 Examples:

In this section two examples are presented. Both examples illustrate the fault analysis algorithm discussed in the section 5.3. The second example also illustrates how stuck-on-one faults are modeled using Affine Sequential Circuits.

Example 5.2:

Consider the ASC shown in Figure 5.2 defined over GF(2).

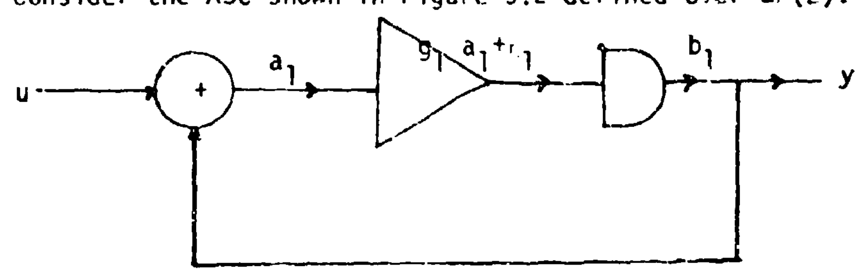

Figure 5.2: Figure for Example 5.2

Assume that the impulse response of the linear part of an ASC is measured and is given by $h_0 = 0$, $h_1 = 1$, $h_2 = 1$, $h_3 = 1$, ... The impulse response of the Affine Sequential Circuit is measured and is given by $h'_0 = 0$, $h'_1 = 0$, $h'_2 = 1$, $h'_3 = 0$, $h'_4 = 1$, $h'_5 = 0$,........ It is desired to compute the values of g_1 and η_1 that are compatible with the given data.

Solution:

The connection equations for this circuit are as follows:

$$\left[\begin{array}{c} a_1 \\ \hline y \end{array}\right] = \left[\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right] \left[\begin{array}{c} b_1 \\ \hline u \end{array}\right] \tag{5.15}$$

and

$$b_1(D) = [g_1/D]\, a_1(D) + \left(\frac{D}{D-1}\right) [1/D]\, \eta_1 \tag{5.16}$$

$$S(D) = [0 + 1(1 - g_1/D \cdot 1)^{-1}\, g_1/D \cdot 1]$$

$$= (1 - g_1/D)^{-1}\, g_1/D \tag{5.17}$$

$$T(D) = \left(\frac{D}{D-1}\right) [1 \cdot (1 - g_1/D \cdot 1)^{-1}(1/D)]$$

$$= (1 - g_1/D)^{-1}\left(\frac{1}{D-1}\right) \tag{5.18}$$

Calculating g_1:

Choose an element $e_1 = \alpha$ from $GF(2^2)$ where α is an indeterminant. Then use (2.25) to compute $S(e_1) = S(\alpha)$ from the given impulse response of the linear part of an ASC.

From (2.25) one obtains,

$$S(\alpha)\alpha^1 = b_{-1} + 1 \tag{5.19}$$

$$S(\alpha)\alpha^2 = b_{-1} + \alpha + 1 \tag{5.20}$$

Simultaneous solution of (5.19) and (5.20) yields,

$$S(\alpha) = \alpha \ . \tag{5.21}$$

Since $S(e_1) = S(D)\big|_{D=e_1=\alpha} = S(\alpha)$, one can replace D by α in (5.17) and obtain,

$$S(\alpha) = \left(1 - \frac{g_1}{\alpha}\right)^{-1} \frac{g_1}{\alpha} \tag{5.22}$$

Simultaneous solution of (5.21) and (5.22) yields

$$g_1 = 1.$$

Substitute $g_1 = 1$ in (5.18) to obtain,

$$T(D) = \left(1 - \frac{1}{D}\right)^{-1}\left(\frac{1}{D-1}\right)$$

$$= \left(\frac{D}{D-1}\right) \frac{1}{(D-1)}$$

$$= \frac{D}{(1+D)^2} \qquad \text{[since } -1 = +1 \text{ in GF(2)]}$$

$$= \frac{D}{1+D^2} \tag{5.23}$$

Obtaining $\{w_k\}$:

from (5.14) one obtains,

$$w_o = h_o - h_o' = 0 - 0 = 0$$

$$w_1 = h_1 - h_1' = 1 - 0 = 1$$

$$w_2 = h_2 - h_2' = 1 - 1 = 0$$

$$w_3 = h_3 - h_3' = 1 - 0 = 1$$

so, $\{w_k\} = \{0, 1, 0, 1, \ldots\ldots\}$, such that $w_o = 0$.

$$W(D) = D\{w_k\} = \{0 \cdot D^0 + 1 \cdot D^{-1} + 0 \cdot D^{-2} + 1 \cdot D^{-2} + \ldots\ldots\}$$

$$W(D) = \frac{D}{1+D^2} \tag{5.24}$$

solving for h:

Substituting (5.23) and (5.24) in (5.10), i.e.,

$$W(D) = T(D)n_1$$

i.e.,
$$\frac{D}{1+D^2} = \frac{D}{1+D^2} \cdot n_1$$

which yields $n_1 = 1$. (5.25)

It is interesting to note from Figure 5.2 that b_1 is exactly opposite to a_1 which is a characteristic of the NOT gate. So ASC's also permit modeling of NOT gate for the purpose of fault analysis.

Example 5.3:

Consider the ASC shown in Figure 5.2. Assume that the impulse response of the linear part of the ASC is measured and is given by $h_0 = 0$, $h_1 = 0$, $h_2 = 0$, ... The impulse response of the Affine Sequential Circuit is measured and is given by $h_0' = 0$, $h_1' = 1$, $h_2' = 1$, $h_3' = 1$, ... It is desired to compute the values of g_1 and n_1 that are compatible with the given data.

The connection equations are the same as those of the previous example. That is, they are given by (5.15), (5.16), (5.17) and (5.18).

Calculating g_1:

Choose an element $e_1 = \alpha$ from $GF(2^2)$. Use (2.25) and the impulse response of the linear part of an ASC to compute $S(\alpha)$. This is given by
$$S(\alpha) = 0. \qquad (5.26)$$

Simultaneous solution of (5.26) and (5.22) yields $g_1 = 0$.

Substitute $g_1 = 0$ in (5.18) to obtain,

$$T(D) = \frac{1}{D+1} \qquad (5.27)$$

Obtaining w_k:

$$w_0 = h_0 - h_0' = 0$$

$$w_1 = h_1 - h_1' = 1$$

$$w_2 = h_2 - h_2' = 1$$

so, $\{w_k\} = \{0, 1, 1, 1, 1, \ldots\}$ such that $w_0 = 0$.

$$W(D) = D\{w_k\} = \{0 \cdot D^0 + 1 \cdot D^{-1} + 1 \cdot D^{-2} + 1 \cdot D^{-3} + \ldots\}$$

$$= \frac{1}{D+1} \quad . \tag{5.28}$$

Substituting (5.27) and (5.28) in (5.10) yields $\eta_1 = 1$.

The above example is an illustration of a "stuck-on-one" fault.

CHAPTER VI

CONCLUSION

In this work, a spectral theory for Linear Sequential Circuits has been formulated and the component connection theory has been applied to LSC's. From the spectral theory and the connection function of the component connection theory, a fault analysis procedure for a Linear Sequential Circuit has been developed. This procedure parallels the multifrequency testing technique for fault analysis in analog circuits. It has been shown that it is often easier to do fault analysis in sequential circuits than in combinatorial circuits.

This fault analysis algorithm has been extended to include Affine Sequential Circuits. The fault analysis procedure for Affine Sequential Circuits has turned out to be no harder than fault analysis in LSC's (even though ASC's are nonlinear). It has been shown with the help of an example that Affine Sequential Circuits can be used for modeling nonlinear faults such as "stuck-on-one" faults. Affine Sequential Circuits should cover a broad range of digital circuits hence an interesting area for further research.

Not much work is done in the area of fault analysis in nonlinear analog or digital circuits. Due to the finiteness of states of sequential circuits, one may possibly generate fault analysis algorithms for nonlinear sequential circuits. Such algorithms can then be generalized to include large scale digital systems.

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APPENDIX A

GALOIS FIELD THEORY

Finite Fields : A finite field [8,10] is a field with a finite number of elements. These fields are known as Galois Fields in honor of French mathematician who first investigated their properties.

Let I denote the integral domain of all integers and p be a prime number. Consider the system I/(p), whose elements are,

$$\overline{0} = (0 + B_m)$$

$$\overline{1} = (1 + B_m)$$

$$\vdots$$

$$\overline{p-1} = (p-1 + B_m)$$

where $B_m = \{ b \mid b=kp,\ p \in I,\ k=0,1,2,3,\ldots\ldots\ldots\ldots \}$

If $\overline{a}$ and $\overline{c}$ are elements of I/(p), addition and multiplication are given by,

$$\overline{a} + \overline{c} = \overline{a + c}$$

$$\overline{a}\,\overline{c} = \overline{a\,c}$$

It is known that I/(p) forms a finite field. This field is denoted by GF(p). For example, GF(2) has two elements. They are $\overline{0}$ and $\overline{1}$. It's addition and multiplication tables are given below :

+	$\overline{0}$	$\overline{1}$
$\overline{0}$	$\overline{0}$	$\overline{1}$
$\overline{1}$	$\overline{1}$	$\overline{0}$

.	$\overline{0}$	$\overline{1}$
$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$

Extension Fields : If E is a field and F is a subfield of E, E is an extension of F. The relation of being an extension of F is denoted by F E.

The following theorems characterize the properties of an extension field. Their proofs are given in refernces [13].

Theorem 1 : Each Galois field contains a unique subfield with a prime number of elements.

Theorem 2 : For every positive integer m and a prime number p, there exists an irreducible polynomial in GF(p) of degree m and these fields, denoted by $GF(p^m)$, have p^m elements. $GF(p^m)$ is an finite extension of F.

Kronecker Theorem : If f(x) is a polynomial with coefficients in a field F, there exists an extension F of F such that for some x_0 in F, $f(x_0) = 0$.

The algorithm for generating elements of $GF(p^m)$ is described below:

Pick an irreducible polynomial P(x) of degree m. Introduce a new symbol, α, and assume $P(\alpha) = 0$. Then $0, 1, \ldots\ldots, \alpha^{p^m - p}$ will be a set of p^m elements of $GF(p^m)$ such that

(i) $\alpha^{p^m - p + 1} = 1$

(ii) $\alpha^i \alpha^j = \alpha^{(i+j) \bmod p^m - p + 1}$

APPENDIX B

DEFINITION OF THE D-TRANSFORM

Let I denote natural integers and G be the set of all sequences. Let $g : I \rightarrow GF(p^m)$ for all $g \in G$.

The D-transform of g(I) is defined in terms of formal power series in indeteminant D denoted by $D\{g(i)\}$.

$$D\{g(i)\} = \sum_{i=-\infty}^{i=\infty} g(i)\, D^{-i}$$

$$= \ldots+\ldots+\ldots+D^3 g(-3) + D^2 g(-2) + Dg(-1) + g(0) + D^{-1} g(1) + D^{-2} g(2) +\ldots+\ldots+D^{-k} g(k) +\ldots+\ldots+\ldots+\ldots$$